ON AN EIGENVALUE PROBLEM INVOLVING THE FRACTIONAL $(s,p)$-LAPLACIAN

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Abstract

In this paper we analyze an eigenvalue problem involving the fractional $(s,p)$-Laplacian, which possesses on the one hand a continuous family of eigenvalues and, on the other hand, one more eigenvalue, which is isolated in the set of eigenvalues of the problem.

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1. Introduction

In recent years increasing attention has been paid to the study of differential and partial differential equations involving nonlocal operators, especially fractional Laplacian-type operators. The interest in studying such problems was stimulated by their applications in continuum mechanics, phase transition phenomena, population dynamics, image processing and game theory, see [2, 4, 11, 13] and the references therein. In this paper, we study some eigenvalue problems associated to a nonlinear version of the fractional Laplacian, the fractional $(s,p)$-Laplacian, for $0 < s < 1$ and $1 < p < \infty$.

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain with Lipschitz boundary $\partial \Omega$. For each $p \in (1, \infty)$ and $s \in (0,1)$, we define the nonlocal nonlinear operator, called the fractional $(s,p)$-Laplacian, that is

$$(-\Delta_p)^s u(x) := 2 \text{ p.v.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+sp}} \, dy, \quad x \in \mathbb{R}^N. \quad (1.1)$$
The common eigenvalue problem associated with this operator is given by
\[
\begin{cases}
(−\Delta_p)^s u(x) = \lambda |u(x)|^{p-2}u(x), & x \in \Omega \\
u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega.
\end{cases}
\] (1.2)

Different aspects concerning problem (1.2) have been investigated over time by Franzina & Palatucci [10], Lindgren & Lindqvist [14], Brasco, Parini & Squassina [3], Del Pezzo & Quass [6], Del Pezzo, Fernandez Bonder & Lopez Rios [5] and Fărășeanu, Mihăilescu and Stancu-Dumitru [9]. It is well known that the first eigenvalue of (1.2), denoted by \(\lambda_1(s,p)\), is positive, simple and isolated (see e.g. [6, Theorems 4.9 and 4.11] and [14, Theorems 14 and 19]). Moreover, its associated eigenfunctions never change signs in the domain \(\Omega\).

In this paper, we are concerned with the following eigenvalue problem
\[
\begin{cases}
(−\Delta_p)^s u(x) = \lambda f(x,u(x)), & x \in \Omega \\
u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega,
\end{cases}
\] (1.3)

where \(\lambda\) is a real number and \(f : \Omega \times \mathbb{R} \to \mathbb{R}\) is given by
\[
f(x,t) = \begin{cases}
h(x,t), & \text{if } t \geq 0, \\
|t|^{p-2}t, & \text{if } t < 0.
\end{cases}
\] (1.4)

We assume that \(h : \Omega \times [0, \infty) \to \mathbb{R}\) is a Carathéodory function satisfying the following hypotheses

\textbf{(H1):} there exists a positive constant \(C \in (0,1)\) such that \(|h(x,t)| \leq C t^{p-1}\), for any \(t \geq 0\) and a.e. \(x \in \Omega\);

\textbf{(H2):} there exists \(t_0 > 0\) such that \(H(x,t_0) = \int_0^{t_0} h(x,s) \, ds > 0\) for a.e. \(x \in \Omega\);

\textbf{(H3):} \(\lim_{t \to \infty} \frac{h(x,t)}{t^{p-1}} = 0\), uniformly in \(x\).

Examples of functions \(h\) which satisfy hypotheses (H1)-(H3) are given in [15], but we recall them here for readers’ convenience:

1. \(h(x,t) = \sin(t/k)\), for any \(t \geq 0\) and any \(x \in \Omega\), where \(k > 1\) is a constant;
2. \(h(x,t) = t \log(1 + t)\), for any \(t \geq 0\) and any \(x \in \Omega\), where \(k \in (0,1)\) is a constant;
3. \(h(x,t) = g(x)(t^q(x)-1) - t^r(x)-1\), for any \(t \geq 0\) and any \(x \in \Omega\), where \(q, r : \Omega \to (1,p)\) are continuous functions satisfying \(\max_{\Omega} r < \min_{\Omega} q\) and \(g \in L_\infty(\Omega)\) satisfies \(0 < \inf_{\Omega} g \leq \sup_{\Omega} g < 1\).

Note that, in the case when the fractional \((s,p)\)-Laplacian from the left-hand side of equation (1.3), is replaced by different type of fractional operators, equations of type (1.3), were studied in [11] and [18]. Also, the results from this paper complement those obtained by Mihăilescu & Rădulescu [15] and Pucci & Rădulescu [16] in the case when in the left side of equation

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The rest of the paper is organized as follows: in Section 2 we introduce the functional space setting where problem (1.3) will be analyzed and we present the main result of the paper; Section 3 is devoted to the proof of the main result of the paper.

2. Main result

We start this section by introducing the natural function space setting where we will analyze problem (1.3). For more details we refer the reader to the book [12] and to the papers [3, 5, 6, 8].

Following the lines from [3, p. 1814] we recall that the natural setting for equations involving the operator \(-\Delta p\) is the fractional Sobolev space \(W^{s,p}_0(\mathbb{R}^N)\) defined as the completion of \(C_0^\infty(\mathbb{R}^N)\) with respect to the Gagliardo seminorm defined as

\[
[u]_{W^{s,p}(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}}.
\]

Taking into account the Dirichlet condition \(u = 0\) in \(\mathbb{R}^N \setminus \Omega\) we consider the space

\[\widetilde{W}^{s,p}_0(\Omega) := \left\{ u : \mathbb{R}^N \to \mathbb{R} : [u]_{W^{s,p}(\mathbb{R}^N)} < \infty \text{ and } u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\},\]

endowed with \([\cdot]_{W^{s,p}(\mathbb{R}^N)}\). This is a reflexive Banach space. Moreover, since \(\Omega\) is a bounded domain with Lipschitz boundary it is well known (see, e.g. [3, Proposition B.1]) that \(\widetilde{W}^{s,p}_0(\Omega)\) coincides with the completion of \(C_0^\infty(\Omega)\) with respect to the norm \([\cdot]_{W^{s,p}(\mathbb{R}^N)}\). Furthermore, we recall that \(\widetilde{W}^{s,p}_0(\Omega)\) is compactly embedded in \(L^q(\Omega)\) for each real number \(q \in [1,p]\) (see, e.g. [8, Theorem 7.1]).

Using these definitions and notations, the first eigenvalue of problem (1.2), \(\lambda_1(s,p)\), can be characterized from a variational point of view by

\[
\lambda_1(s,p) := \inf_{u \in \widetilde{W}^{s,p}_0(\Omega) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |u|^p \, dx}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy},
\]

(2.1)

We define an eigenvalue of (1.3) as being a real number \(\lambda\) for which there exists a function \(u \in \widetilde{W}^{s,p}_0(\Omega) \setminus \{0\}\) such that

\[
E_{s,p}(u,v) = \lambda \int_{\Omega} f(x,u)v(x) \, dx, \quad \forall \, v \in \widetilde{W}^{s,p}_0(\Omega),
\]

(2.2)
where

\[ E_{s,p}(u, v) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} \, dx \, dy, \]

for all \( u, v \in \widetilde{W}^{s,p}_0(\Omega) \).

The main result of this paper is given by the following theorem.

**Theorem 2.1.** Assume that \( f \) is given by relation (1.4) and conditions (H1), (H2) and (H3) are fulfilled. Then \( \lambda_1(s, p) \) defined in (2.1) is an isolated eigenvalue of problem (1.3). Moreover, any \( \lambda \in (0, \lambda_1(s, p)) \) is not an eigenvalue of problem (1.3), but there exists \( \mu_1 > \lambda_1(s, p) \), such that any \( \lambda \in (\mu_1, \infty) \) is an eigenvalue of problem (1.3).

3. Proof of main result

For each \( u \in \widetilde{W}^{s,p}_0(\Omega) \) we set

\[ u_\pm(x) = \max\{\pm u(x), 0\}, \quad \forall x \in \Omega. \]

By [7, Lemma 2.2] we have that \( u_+, u_- \in \widetilde{W}^{s,p}_0(\Omega) \) while by [6, (4.29)] the following estimates hold true

\[ |u(x) - u(y)|^{p-2}(u(x) - u(y))(u_+(x) - u_+(y)) \geq |u_+(x) - u_+(y)|^p, \quad (3.1) \]

and

\[ -|u(x) - u(y)|^{p-2}(u(x) - u(y))(u_-(x) - u_-(y)) \geq |u_-(x) - u_-(y)|^p, \quad (3.2) \]

for all \( x, y \in \Omega \). Thus, problem (1.3) with \( f \) given by relation (1.4) becomes

\[ \begin{cases} \begin{align*} (-\Delta_p)^s u(x) &= \lambda (h(x, u_+) - u_-^{p-1}), \quad x \in \Omega, \\ u(x) &= 0, \quad \text{for } x \in \mathbb{R}^N \setminus \Omega. \end{align*} \end{cases} \quad (3.3) \]

We define an eigenvalue of (3.3), a real number \( \lambda \), such that there exists \( u \in \widetilde{W}^{s,p}_0(\Omega) \setminus \{0\} \) a weak solution of problem (1.3), i.e.

\[ E_{s,p}(u, v) = \lambda \int_{\Omega} \left( h(x, u_+) - u_-^{p-1} \right) v \, dx, \quad \forall v \in \widetilde{W}^{s,p}_0(\Omega). \quad (3.4) \]

**Lemma 3.1.** Any \( \lambda \in (0, \lambda_1(s, p)) \) is not an eigenvalue of problem (3.3).

**Proof.** Assume that \( \lambda > 0 \) is an eigenvalue of problem (3.3) with the corresponding eigenfunction \( u \). Taking \( v = u_+ \) and \( v = u_- \) in (3.4) we obtain

\[ E_{s,p}(u, u_+) = \lambda \int_{\Omega} h(x, u_+)u_+ \, dx \quad (3.5) \]
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and

\[ E_{s,p}(u, u) = -\lambda \int_{\Omega} u^p \, dx. \]  (3.6)

By the definition of \( \lambda_1(s, p) \), we have

\[ \lambda_1(s, p) \int_{\Omega} |v|^p \, dx \leq \|v\|_{W^{s,p}(\mathbb{R}^N)}^p = E_{s,p}(v, v), \quad \forall \, v \in \tilde{W}_0^{s,p}(\Omega). \]  (3.7)

Taking into account relations (3.5), (3.7), (3.1), and hypothesis (H1) we have

\[ \lambda_1(s, p) \int_{\Omega} u^p \, dx \leq \|u\|_{W^{s,p}(\mathbb{R}^N)}^p = E_{s,p}(u, u) = \lambda \int_{\Omega} h(x, u_+) u_+ \, dx \]
\[ \leq \lambda \int_{\Omega} u^p_+ \, dx, \]

while by relations (3.6), (3.7) and (3.2) we deduce

\[ \lambda_1(s, p) \int_{\Omega} u^p_+ \, dx \leq \|u_+\|_{W^{s,p}(\mathbb{R}^N)}^p = E_{s,p}(u, u) = \lambda \int_{\Omega} u^p_+ \, dx, \]

Since \( \lambda \) is an eigenvalue of problem (3.3), then \( u \neq 0 \) and by the above pieces of information we deduce that \( u_+ \neq 0 \) or \( u_- \neq 0 \). Hence, the last two inequalities show that \( \lambda \) is an eigenvalue of problem (3.3) only if \( \lambda_1(s, p) \leq \lambda \).

**Lemma 3.2.** The first eigenvalue of (1.2), \( \lambda_1(s, p) \), is also an eigenvalue of problem (3.3).

**Proof.** As we have already pointed out, \( \lambda_1(s, p) \) is the lowest eigenvalue of problem (1.2), it is simple and the corresponding eigenfunctions do not change sign in domain \( \Omega \). Then, there exists \( u_1 \in \tilde{W}_0^{s,p}(\Omega) \backslash \{0\} \) with \( u_1(x) \leq 0 \), for each \( x \in \Omega \), an eigenfunction corresponding to \( \lambda_1(s, p) \), i.e.

\[ E_{s,p}(u_1, v) = \lambda_1(s, p) \int_{\Omega} |u_1|^{p-2} u_1 v \, dx = -\lambda_1(s, p) \int_{\Omega} (-u_1)^{p-1} v \, dx, \]

for any \( v \in \tilde{W}_0^{s,p}(\Omega) \). Hence, we have \( (u_1)_+ = 0 \) and \( (u_1)_- = -u_1 \) and we deduce that relation (3.4) holds true with \( u = u_1 \) and \( \lambda = \lambda_1(s, p) \). Consequently, \( \lambda_1(s, p) \) is an eigenvalue of problem (3.3). The proof of Lemma 3.2 is complete.

**Lemma 3.3.** \( \lambda_1(s, p) \) is an isolated eigenvalue of problem (3.3).

**Proof.** By Lemma 3.1, we have that \( \lambda_1(s, p) \) is isolated in a neighborhood to the left. We will prove that it is also isolated in a neighborhood
to the right. To this aim, let \( \lambda \geq \lambda_1(s, p) \) be an eigenvalue of problem (3.3) with a corresponding eigenfunction \( u \in \tilde{W}^{s,p}_0(\Omega) \). If its corresponding positive part, that is \( u_+ \), is not identically zero, then by (3.7), (3.1) and hypothesis (H1) we deduce

\[
\lambda_1(s, p) \int_\Omega u_+^p \, dx \leq [u_+]_p^{W^{s,p}(\mathbb{R}^N)} \leq E_{s,p}(u, u_+) = \lambda \int_\Omega h(x, u_+)u_+ \, dx \\
\leq \lambda C \int_\Omega u_+^p \, dx.
\]

Since \( C \in (0, 1) \), then \( \lambda_1(s, p) < \lambda_1(s, p)/C \leq \lambda \). It follows that, if \( \lambda \in (0, \lambda_1(s, p)/C) \) is an eigenvalue of problem (3.3), then it has a corresponding eigenfunction \( u \in \tilde{W}^{s,p}_0(\Omega) \) with \( u \leq 0 \) in \( \Omega \), or

\[
E_{s,p}(u, v) = -\lambda \int_\Omega (u)^{p-1}v \, dx = \lambda \int_\Omega |u|^{p-2}uv \, dx, \quad \forall \, v \in \tilde{W}^{s,p}_0(\Omega).
\]

It means that \( \lambda \) is an eigenvalue of problem (1.2), too. But we have already noted that \( \lambda_1(s, p) \) is an isolated eigenvalue of problem (1.2), i.e. there exists \( \epsilon > 0 \) such that in the interval \( (\lambda_1(s, p), \lambda_1(s, p) + \epsilon) \) there is no eigenvalue of problem (1.2). Thus, taking \( \delta := \min\{\lambda_1(s, p)/C, \lambda_1(s, p) + \epsilon\} \), we observe that \( \delta > \lambda_1(s, p) \) and any \( \lambda \in (\lambda_1(s, p), \delta) \) cannot be an eigenvalue of problem (1.2), and consequently any \( \lambda \in (\lambda_1(s, p), \delta) \) is not an eigenvalue of problem (3.3). We conclude that \( \lambda_1(s, p) \) is an isolated eigenvalue in the set of eigenvalues of problem (3.3). The proof of Lemma 3.3 is complete.

In the following, we will show that there exists \( \mu_1 > 0 \) such that any \( \lambda \in (\mu_1, \infty) \) is an eigenvalue of problem (3.3). In order to do this, we consider the eigenvalue problem

\[
\begin{align*}
\left\{ \begin{array}{l}
(-\Delta)^s_p u(x) = \lambda h(x, u_+), \quad x \in \Omega \\
u(x) = 0, \quad \text{for} \, x \in \mathbb{R}^N \setminus \Omega.
\end{array} \right.
\end{align*}
\]

(3.8)

We say that a real number \( \lambda \) is an eigenvalue of (3.8) if there exists \( u \in \tilde{W}^{s,p}_0(\Omega) \setminus \{0\} \) such that

\[
E_{s,p}(u, v) = \lambda \int_\Omega h(x, u_+)v(x) \, dx, \quad \forall \, v \in \tilde{W}^{s,p}_0(\Omega).
\]

(3.9)

We note that if \( \lambda \) is an eigenvalue for (3.8) with the corresponding eigenfunction \( u \), then testing with \( v = u_- \) in the above relation we deduce

\[
E_{s,p}(u, u_-) = 0,
\]

or, by relation (3.2) we have that

\[
[u_-]_p^{W^{s,p}(\mathbb{R}^N)} \leq -E_{s,p}(u, u_-) = 0
\]
which implies \( u^- = 0 \). Thus, we find \( u \geq 0 \). In other words, the eigenvalues of problem (3.8) possess only nonnegative corresponding eigenfunctions. Moreover, by the above facts, we deduce that an eigenvalue of problem (3.8) is an eigenvalue of problem (3.3).

For each \( \lambda > 0 \) we define the energy functional associated to problem (3.8) by

\[
J_\lambda : W^{s,p}_0(\Omega) \to \mathbb{R},
\]

\[
J_\lambda (u) := \frac{1}{p} [u]^{p}_{W^{s,p}(\mathbb{R}^N)} - \lambda \int_\Omega H(x, u_+) \, dx,
\]

where \( H(x, t) = \int_0^t h(x, s) \, ds \).

Standard arguments show that \( J_\lambda \in C^1(\widetilde{W}^{s,p}_0(\Omega), \mathbb{R}) \) with the derivative given by

\[
\langle J_\lambda'(u), v \rangle = E_{s,p}(u,v) - \lambda \int_\Omega h(x, u_+ v) \, dx,
\]

for any \( u, v \in \widetilde{W}^{s,p}_0(\Omega) \). Observe that, in this context, \( \lambda > 0 \) is an eigenvalue of problem (3.8) if and only if there exists a nontrivial critical point of functional \( J_\lambda \).

**Lemma 3.4.** The functional \( J_\lambda \) is bounded from below and coercive.

**Proof.** By hypothesis (H3) we deduce that

\[
\lim_{t \to \infty} \frac{H(x, t)}{t^p} = 0, \quad \text{uniformly in } \Omega.
\]

Then, for a given \( \lambda > 0 \) there exists a positive constant \( C_\lambda > 0 \), such that

\[
\lambda H(x, t) \leq \frac{\lambda_1(s,p)}{2p} t^p + C_\lambda, \quad \forall \ t \geq 0, \ \text{a.e. } x \in \Omega.
\]

Then, for each \( u \in \widetilde{W}^{s,p}_0(\Omega) \) we have

\[
J_\lambda(u) \geq \frac{1}{p} [u]^{p}_{W^{s,p}(\mathbb{R}^N)} - \frac{\lambda_1(s,p)}{2p} \int_\Omega u^p \, dx - C_\lambda |\Omega|
\]

\[
\geq \frac{1}{2p} [u]^{p}_{W^{s,p}(\mathbb{R}^N)} - C_\lambda |\Omega|.
\]

The last inequality shows that \( J_\lambda \) is bounded from below and coercive. The proof of Lemma 3.4 is complete.

**Lemma 3.5.** There exists \( \mu_1 > 0 \) such that, assuming \( \lambda > \mu_1 \), we have \( \inf_{\widetilde{W}^{s,p}_0(\Omega)} J_\lambda < 0 \).
Proof. By hypothesis (H2), we deduce that there exists $t_0 > 0$ such that $H(x, t_0) > 0$ for a.e. $x \in \Omega$. Let $\Omega_1 \subset \Omega$ be a compact subset, sufficiently large, and $u_0 \in C_0^1(\Omega) \subset \overline{W}^{1,p}_0(\Omega)$ such that $u_0(x) = t_0$, for any $x \in \Omega_1$ and $0 \leq u_0(x) \leq t_0$, for any $x \in \Omega \setminus \Omega_1$.

Hence, by hypothesis (H1) we deduce

$$
\int_{\Omega} H(x, u_0) \, dx \geq \int_{\Omega_1} H(x, t_0) \, dx - \int_{\Omega \setminus \Omega_1} C u_0^p \, dx
$$

$$
\geq \int_{\Omega_1} H(x, t_0) \, dx - C t_0^p |\Omega \setminus \Omega_1| > 0.
$$

Then, we infer that

$$
J_\lambda(u_0) \leq \frac{1}{p} [u_0]_{W^{s,p}(\mathbb{R}^N)}^p - \lambda \left( \int_{\Omega_1} H(x, t_0) \, dx - C t_0^p |\Omega \setminus \Omega_1| \right) < 0,
$$

for each

$$
\lambda > \frac{\frac{1}{p} [u_0]_{W^{s,p}(\mathbb{R}^N)}^p}{\int_{\Omega_1} H(x, t_0) \, dx - C t_0^p |\Omega \setminus \Omega_1|}.
$$

Hence, we conclude that there exists $\mu_1 > 0$ such that for any $\lambda > \mu_1$ we have $\inf_{\overline{W}^{s,p}(\Omega)} J_\lambda < 0$. The proof of Lemma 3.5 is complete.

Taking into account Lemmas 3.4 and 3.5 and the fact that $J_\lambda$ is weakly lower semi-continuous, by [17, Theorem 1.2] we deduce that there exists a constant $\mu_1 > 0$ such that $J_\lambda$ possesses a negative global minimum for each $\lambda > \mu_1$. It means that such a $\lambda$ is an eigenvalue of problem (3.8) and consequently an eigenvalue of problem (3.3). Combining these pieces of information with the results of Lemmas 3.1, 3.2 and 3.3, we conclude that Theorem 2.1 holds true.

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